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Flow of a Liquid with Axial Symmetry: Two Examples of Solutions by a New Method

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Submitted by J. M. Richardson

1. In two-dimensional irrotational motion, it is often possible to solve a given problem, by relating it, through a conformal transformation, to another problem whose solution can be effected. In the case of axially symmetric motion, the device of conformal transformation is not, in general, available.

I have recently shown [1] that in axially symmetric, irrotational motion, there is only one nontrivial contact transformation which leaves unchanged the form of the differential equation satisfied by the velocity potential, and one which leaves unchanged the form of the equation satisfied by the Stokes' stream function. The former is precisely Kelvin's transformation: the latter, involving, like Kelvin's, geometrical inversion with respect to a point on the axis of symmetry, is as follows. Let (r, θ) be polar coordinates ($\theta = 0$ being the positive direction of the axis of symmetry), and let ψ be Stokes' stream function for a certain irrotational motion. Let R, Θ, Ψ be related to r, θ, ψ by the transformation

$$\Theta = \theta, \qquad Rr = k^2, \qquad \Psi = -Ck\psi/r \tag{1}$$

where k is the (nonzero) radius of inversion, and C is a nonzero dimensionless constant. Then Ψ is the Stokes' stream function of an axially symmetric, irrotational motion, of which (R, Θ) are polar coordinates. Owing to the nature of the hydrodynamical boundary conditions, it is often more convenient to work with the Stokes' stream function than with the velocity potential. Accordingly, the transformation (1) can sometimes be used to effect the solution.

2. To illustrate this method, we shall solve, by means of the transformation (1), two problems of classical hydrodynamics, namely the flow of a non-viscous liquid past two spheres, with uniform streaming at infinity parallel to the line of centres, in the two cases, namely when the spheres are not in contact, and when they are in contact.

There is, of course, no special interest attached to these problems for their own sake. The former has already been solved approximately, using the velocity potential [2]. The interest lies purely in the use of a new method of solution.

3. Firstly, let the two spheres be of radii a and b respectively, and let the distance between their centers be c ($c > a + b$). Let the velocity at infinity be U, in the positive direction of the axis. See Fig. 1. We taken the origin at

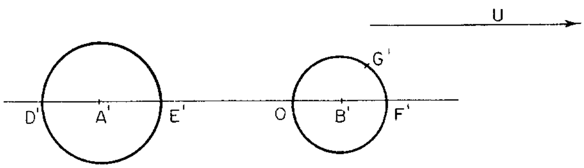

FIG. 1

the point 0, $0F'$ to be $\theta = 0$, $0D'$ to be $\theta = \pi$. Then the boundary conditions are as follows: (i) $\psi = 0$ when $\theta = 0$ and when $\theta = \pi$. (ii) $\psi = 0$ on the surface of both spheres. (iii) $\psi = -\frac{1}{2}Ur^2 \sin^2\theta + O(\log r)$ when $r \to \infty$.

We now make the transformation (1): and, for convenience, we take

$$k^2 = (c - b)^2 - a^2,$$
$$C = -1,$$

so that the transformation is

$$\Theta = \theta, R = [(c-b)^2 - a^2]/r, \Psi = \{\sqrt{[(c-b)^2 - a^2]}\}\psi/r,$$

or (2)

$$\theta = \Theta, r = [(c-b)^2 - a^2]/R, \psi = \{\sqrt{[(c-b)^2 - a^2]}\}\Psi/R.$$

Owing to our choice of the radius of inversion k, it is evident that the left-hand sphere inverts into itself, the transformed figure being as shown in Fig. 2, where corresponding letters indicate corresponding points in the two figures (except that, of course, A is not the inverse of A'). The right-hand

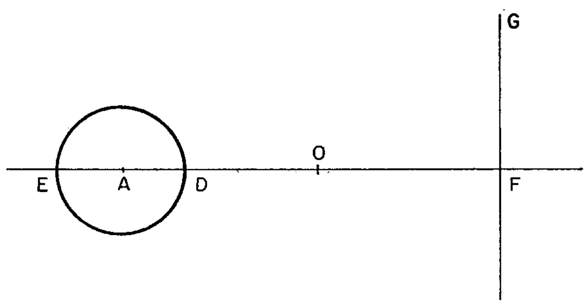

FIG. 2

sphere inverts into the plane FG through F, perpendicular to the axis: and we have

$$EO = \frac{(c-b)^2 - a^2}{(c-b) - a} = (c-b) + a,$$

$$DO = \frac{(c-b)^2 - a^2}{(c-b) + a} = (c-b) - a,$$

and thus

$$AD = a,$$

$$A0 = (c-b).$$

We also have

$$AF = (c-b) + \frac{(c-b)^2 - a^2}{2b}$$

$$= \frac{c^2 - a^2 - b^2}{2b}.$$

The boundary conditions of the transformed problem are thus as follows. (i) $\Psi = 0$ when $\Theta = 0$ or $\Theta = \pi$. (ii) $\Psi = 0$ both on the surface of the sphere ED and on the plane FG. (iii)

$$\Psi = -\tfrac{1}{2}[(c-b)^2 - a^2]^{3/2}\, U \frac{\sin^2\Theta}{R} + O\left\{R \log\left(\frac{1}{R}\right)\right\} \tag{3}$$

when $R \to 0$. But the first term on the right hand side of (3) is precisely the Stokes' stream function for a doublet of strength

$$\tfrac{1}{2}[(c-b)^2 - a^2]^{3/2}U$$

placed at the origin. If we can find the stream function for such a doublet, in the presence of the sphere ED and the plane FG, satisfying conditions (i) and (ii) above, by means of image doublets placed on the axis of symmetry, the problem will be solved. For the stream function due to a series of doublets of strength M_i ($i = 1, 2, \ldots$), situated on the axis at distances h_i ($i = 1, 2, \ldots$) in the positive direction from the origin, is (each $h_i \neq 0$)

$$-R^2 \sin^2\Theta \sum_{i=1}^{\infty} \frac{M_i}{(R^2 - 2h_i R\cos\Theta + h_i^2)^{3/2}} = O(R^2)$$

when $R \to 0$, provided that the series is uniformly convergent. Thus since $O(R^2)$ is certainly $O\{R\log(1/R)\}$ for small R, it follows that the doublet at the origin, and the image system, satisfy the boundary conditions (3) above, subject, of course, to the uniform convergence of the series.

To solve the image problem, we first consider the positions of the images. The image of the doublet at 0, in the sphere, center A, is a doublet at the inverse point: while the image in the plane is a doublet situated at a distance equal to $0F$, to the right of F. These two images give rise to two further images, in the sphere and the plane respectively, and so on; and each successive pair of image doublets is oriented in the opposite direction to the previous pair. Thus let the successive images in the sphere be doublets of strength $-X_0, +X_1, -X_2, \dots (-1)^{n+1}X_n, \dots,$ situated respectively at distances $x_0, x_1, \dots, x_n, \dots$ to the right of A; and let the successive images in the plane be doublets of strength $-Y_0, +Y_1, \dots (-1)^{n+1}Y_n \dots,$ situated respectively at distances $y_0, y_1, \dots y_n, \dots$ to the right of A, each X and Y being positive. Then $-X_0$ is the image in the sphere of the doublet at 0; $-Y_0$ is the image in the plane of the doublet at 0; $(-1)^{n+2}X_{n+1}$ is the image in the sphere of $(-1)^{n+1}Y_n$; $(-1)^{n+2}Y_{n+1}$ is the image in the plane of $(-1)^{n+1}X_n$. Thus we have

$$\left.\begin{aligned}
x_0 &= \frac{a^2}{c-b} \\
y_0 &= \frac{c^2-a^2-b^2}{b} - (c-b) = \frac{c^2-a^2-bc}{b} \\
x_{n+1} &= \frac{a^2}{y_n} \\
y_{n+1} &= \frac{c^2-a^2-b^2}{b} - x_n
\end{aligned}\right\} \tag{4}$$

From (4) we have

$$x_{n+2} = \frac{a^2}{[(c^2-a^2-b^2)/b] - x_n}$$

or

$$x_{n+2}x_n - \frac{(c^2-a^2-b^2)}{b}x_{n+2} + a^2 = 0, \tag{5}$$

and

$$y_{n+2} = \frac{c^2-a^2-b^2}{b} - \frac{a^2}{y_n}$$

or

$$y_{n+2}y_n - \frac{(c^2-a^2-b^2)}{b}y_n + a^2 = 0. \tag{6}$$

It is convenient to solve both (5) and (6) for the even terms only, deducing the expressions for the odd terms from (4). Thus let

$$\begin{aligned}
x_{2n} &= u_n, \qquad n \geqslant 0, \\
y_{2n} &= v_n, \qquad n \geqslant 0.
\end{aligned} \tag{7}$$

Then from (4) we have

$$\begin{aligned} x_{2n+1} &= \frac{c^2 - a^2 - b^2}{b} - y_{2n+2} \\ y_{2n+1} &= \frac{c^2 - a^2 - b^2}{b} - x_{2n} . \end{aligned} \tag{8}$$

Writing $2n$ for n in (5) and (6), and using (7), the problem is thus reduced to that of solving the two equations

$$\left.\begin{aligned} &u_{n+1}u_n - \left(\frac{c^2 - a^2 - b^2}{b}\right) u_{n+1} + a^2 = 0, \\ &v_{n+1}v_n - \left(\frac{c^2 - a^2 - b^2}{b}\right) v_n + a^2 = 0, \\ &\text{subject to the initial conditions} \\ &u_0 = x_0 = a^2/(c - b) \\ &v_0 = y_0 = [(c^2 - a^2 - b^2)/b] - (c - b). \end{aligned}\right\} \tag{9}$$

To simplify the solution of (9), we change the constants a little. Let

$$c^2 - a^2 - b^2 = ab(x + 1/x), \tag{10}$$

and let $0 < x \leqslant 1$ (which is clearly permissible), so that

$$x = \frac{(c^2 - a^2 - b^2) - \sqrt{\{[c + (a + b)][c - (a + b)](c + | a - b |)(c - | a - b |)\}}}{2ab} .$$

Then since $c > a + b$, and hence $c^2 - a^2 - b^2 > 2ab$, we have $0 < x < 1$. Again,

$$(c - b)^2 - a^2 = (c - a - b)(c + a - b) > 0.$$

Therefore, using (10),

$$ab(x + 1/x) + 2b^2 - 2bc = 2b[\tfrac{1}{2}a(x + 1/x) - (c - b)] > 0,$$

so that $a < c - b < \frac{1}{2}a(x + 1/x)$. Hence we have

$$\begin{aligned} &a(1 - x) > a - x(c - b) > \tfrac{1}{2}a(1 - x^2) > 0, \\ &\tfrac{1}{2}a(1/x - x) > (c - b) - ax > a(1 - x) > 0. \end{aligned} \tag{11}$$

Let λ be defined by

$$\lambda = x\left(\frac{(c - b) - ax}{a - x(c - b)}\right) \tag{12}$$

Then it follows at once from (11) that $0 < x < \lambda < 1$. From (12) we have at once

$$(c - b) = \frac{a}{x}\frac{\lambda + x^2}{\lambda + 1}$$

and thus (13)

$$[(c - b)^2 - a^2]^{1/2} = \frac{a}{x}\frac{1}{\lambda + 1}[(\lambda^2 - x^2)(1 - x^2)]^{1/2}.$$

Substituting from (10) and (13) in (9), we thus have

$$\left.\begin{aligned} &u_{n+1}u_1 - a(x + 1/x)u_{n+1} + a^2 = 0, \\ &v_{n+1}v_n - a(x + 1/x)v_n + a^2 = 0, \\ &u_0 = ax \cdot \frac{\lambda + 1}{\lambda + x^2} \\ &v_0 = \frac{a}{x} \cdot \frac{1 + \lambda x^2}{1 + \lambda}, \end{aligned}\right\} \tag{14}$$

To solve (14), we first observe that if, for any n, $u_n = ax$, then, from the first equation, $u_{n-1} = ax$, $u_{n-2} = ax$, ..., $u_0 = ax$, contradicting the given value of u_0. Similarly, if, for any n, $v_n = a/x$, then $v_0 = a/x$, contradicting the given value of v_0. Thus we may write

$$u_n = a(x + 1/p_n),$$
$$v_n = a(1/x - 1/q_n),$$

so that, after simplifying, we have

$$-x^2 p_{n+1} + p_n = x,$$
$$-x^2 q_{n+1} + q_n = x,$$

the solutions to which may conveniently be written as

$$p_n = \frac{x}{1 - x^2}(1 + A/x^{2n+2}),$$

$$q_n = \frac{x}{1 - x^2}(1 + B/x^{2n}),$$

A and B being constants. We now have

$$u_n = ax\frac{A + x^{2n}}{A + x^{2n+2}}$$

$$v_n = \frac{a}{x}\frac{B + x^{2n+2}}{B + x^{2n}}.$$

Putting in the initial values (14), we at once obtain $A = \lambda$, $B = 1/\lambda$. Hence we finally have, using (7), (8), (10),

$$\begin{aligned} x_{2n} &= ax \frac{\lambda + x^{2n}}{\lambda + x^{2n+2}}, \\ y_{2n} &= \frac{a}{x} \frac{1 + \lambda x^{2n+2}}{1 + \lambda x^{2n}}, \\ x_{2n+1} &= ax \frac{1 + \lambda x^{2n}}{1 + \lambda x^{2n+2}}, \\ y_{2n+1} &= \frac{a}{x} \frac{\lambda + x^{2n+4}}{\lambda + x^{2n+2}}, \\ n &\geqslant 0. \end{aligned} \tag{15}$$

Next, regarding the strengths of the various doublets, we notice that, since $(-1)^{n+2}X_{n+1}$ is the image in the sphere of $(-1)^{n+1}Y_n$, and $(-1)^{n+2}Y_{n+1}$ the image in the plane of $(-1)^{n+1}X_n$,

$$\begin{aligned} \frac{X_{n+1}}{Y_n} &= \left(\frac{x_{n+1}}{a}\right)^3, \\ Y_{n+1} &= X_n . \end{aligned} \tag{16}$$

Hence

$$\frac{X_{2n}}{X_{2n-2}} = \left(\frac{x_{2n}}{a}\right)^3 = \frac{x^3(\lambda + x^{2n})^3}{(\lambda + x^{2n+2})^3} \qquad (n \geqslant 1),$$

and therefore

$$\frac{X_{2n}}{X_0} = x^{3n} \frac{(\lambda + x^2)^3}{(\lambda + x^{2n+2})^3}.$$

But, the doublet at the origin being of strength (using (13))

$$\tfrac{1}{2}[(c-b)^2 - a^2]^{3/2} U = \frac{\tfrac{1}{2} a^3 U[(\lambda^2 - x^2)(1 - x^2)]^{3/2}}{(1+\lambda)^3 x^3}, \tag{17}$$

we have

$$X_0 = \tfrac{1}{2} a^3 U \frac{[(\lambda^2 - x^2)(1 - x^2)]^{3/2}}{(\lambda + x^2)^3}$$

and thus

$$X_{2n} = \tfrac{1}{2} a^3 U [(\lambda^2 - x^2)(1 - x^2)]^{3/2} \cdot \frac{x^{3n}}{(\lambda + x^{2n+2})^3} \qquad (n \geqslant 0).$$

Again, from (16) we have

$$\frac{Y_{2n}}{Y_{2n-2}} = \left(\frac{x_{2n-1}}{a}\right)^3 = x^3 \frac{(1+\lambda x^{2n-2})^3}{(1+\lambda x^{2n})^3} \qquad (n \geqslant 1)$$

and therefore

$$\frac{Y_{2n}}{Y_0} = x^{3n} \frac{(1+\lambda)^3}{(1+\lambda x^{2n})^3}$$

and thus, since Y_0 is equal in magnitude to the doublet at the origin,

$$Y_{2n} = \tfrac{1}{2} a^3 U[(\lambda^2 - x^2)(1 - x^2)]^{3/2} \frac{x^{3n-3}}{(1+\lambda x^{2n})^3} \qquad (n \geqslant 0).$$

Finally, from (16) we have

$$\begin{aligned} X_{2n+1} &= Y_{2n+2} \\ Y_{2n+1} &= X_{2n} \,. \end{aligned} \qquad (n \geqslant 0)$$

To write down the Stokes' stream function corresponding to each doublet, we require the distance from the origin, 0, rather than from the point A. Let w_i denote the distance of the doublet $(-1)^{i+1}X_i$ from 0, measured to the left, so that w_i is positive, and let z_i be the distance of the doublet $(-1)^{i+1}Y_i$ from 0, measured to the right, so that z_i is also positive. Then we have

$$w_i = (c - b) - x_i = \frac{a}{x}\frac{\lambda + x^2}{\lambda + 1} - x_i \,,$$

$$z_i = y_i - (c - b) = y_i - \frac{a}{x}\frac{\lambda + x^2}{\lambda + 1} \,.$$

From (15), we thus have

$$\begin{aligned} w_{2n} &= \frac{a}{(1+\lambda)} \frac{(1-x^2)}{x} \frac{(\lambda^2 - x^{2n+2})}{(\lambda + x^{2n+2})} \\ w_{2n+1} &= a \frac{\lambda}{1+\lambda} \frac{1-x^2}{x} \frac{(1 - x^{2n+2})}{(1+\lambda x^{2n+2})} \\ z_{2n} &= \frac{a}{1+\lambda} \frac{1-x^2}{x} \frac{(1 - \lambda^2 x^{2n})}{(1+\lambda x^{2n})} \\ z_{2n+1} &= a \frac{\lambda}{1+\lambda} \frac{1-x^2}{x} \frac{(1 - x^{2n+2})}{(\lambda + x^{2n+2})} \end{aligned} \qquad (18)$$

and, recapitulating,

$$
\begin{aligned}
X_{2n} &= \tfrac{1}{2}\, a^3 U[(\lambda^2 - x^2)(1 - x^2)]^{3/2} \frac{x^{3n}}{(\lambda + x^{2n+2})^3}\,, \\
X_{2n+1} &= \tfrac{1}{2}\, a^3 U[(\lambda^2 - x^2)(1 - x^2)]^{3/2} \frac{x^{3n}}{(1 + \lambda x^{2n+2})^3}\,, \\
Y_{2n} &= \tfrac{1}{2}\, a^3 U[(\lambda^2 - x^2)(1 - x^2)]^{3/2} \frac{x^{3n-3}}{(1 + \lambda x^{2n})^3}\,, \\
Y_{2n+1} &= \tfrac{1}{2}\, a^3 U[(\lambda^2 - x^2)(1 - x^2)]^{3/2} \frac{x^{3n}}{(\lambda + x^{2n+2})^3}\,.
\end{aligned}
\tag{19}
$$

Let now R_i denote the distance from the point (R, Θ) to the doublet $(-1)^{i+1}X_i$, and S_i the distance from (R, Θ) to the doublet $(-1)^{i+1}Y_i$. Then we have

$$
R_i^2 = R^2 + 2w_i R \cos \Theta + w_i^2,
$$

$$
S_i^2 = R^2 - 2z_i R \cos \Theta + z_i^2,
$$

and the stream function is, using (17) and (19)

$$
\Psi = -\tfrac{1}{2} a^3 U[(\lambda^2 - x^2)(1 - x^2)]^{3/2} R^2 \sin^2 \Theta
\left[
\begin{aligned}
&\frac{1}{(1 + \lambda)^3 x^3} \frac{1}{R^3} \\
&- \sum_{n=0}^{\infty} \frac{x^{3n}}{(\lambda + x^{2n+2})^3} \frac{1}{R_{2n}^3} \\
&+ \sum_{n=0}^{\infty} \frac{x^{3n}}{(1 + \lambda x^{2n+2})^3} \frac{1}{R_{2n+1}^3} \\
&- \sum_{n=0}^{\infty} \frac{x^{3n-3}}{(1 + \lambda x^{2n})^3} \frac{1}{S_{2n}^3} \\
&+ \sum_{n=0}^{\infty} \frac{x^{3n}}{(\lambda + x^{2n+2})^3} \frac{1}{S_{2n+1}^3}
\end{aligned}
\right].
$$

Now, in any bounded domain of (R, Θ) which lies to the left of the plane FG, and outside the sphere DE, it is evident that each of $1/R_{2n}$, $1/R_{2n+1}$, $1/S_{2n}$, $1/S_{2n+1}$ is bounded: and owing to the factor x^{3n}, we see that each of the four series in the above expression, together with any of the series obtained by partial differentiation with respect to R and Θ, is absolutely and uniformly convergent in such a domain.

Finally, making the transformation (2), back to the variables r, θ, ψ, taking account of (13), we have the solution to the original problem, namely,

$$\psi = -\tfrac{1}{2}Ur^2\sin^2\theta \left[\begin{array}{l} 1 - \sum\limits_{n=0}^{\infty} \dfrac{x^{3n}}{(\lambda + x^{2n+2})^3}\dfrac{1}{\rho_{2n}^3} + \sum\limits_{n=0}^{\infty} \dfrac{x^{3n}}{(1 + \lambda x^{2n+2})^3}\dfrac{1}{\rho_{2n+1}^3} \\ - \sum\limits_{n=0}^{\infty} \dfrac{x^{3n-3}}{(1 + \lambda x^{2n})^3}\dfrac{1}{\sigma_{2n}^3} + \sum\limits_{n=0}^{\infty} \dfrac{x^{3n}}{(\lambda + x^{2n+2})^3}\dfrac{1}{\sigma_{2n+1}^3} \end{array}\right]$$

where

$$\rho_{2n}^2 = \left\{\begin{array}{l} \dfrac{1}{(1+\lambda)^2x^2} + 2\dfrac{1}{(1+\lambda)x}\dfrac{1}{(\lambda^2 - x^2)}\dfrac{(\lambda^2 - x^{2n+2})}{(\lambda + x^{2n+2})}\left(\dfrac{r}{a}\right)\cos\theta \\ \qquad + \dfrac{1}{(\lambda^2 - x^2)^2}\dfrac{(\lambda^2 - x^{2n+2})^2}{(\lambda + x^{2n+2})^2}\left(\dfrac{r}{a}\right)^2, \end{array}\right\},$$

$$\rho_{2n+1}^2 = \left\{\begin{array}{l} \dfrac{1}{(1+\lambda)^2x^2} + 2\dfrac{1}{(1+\lambda)x}\dfrac{\lambda}{(\lambda^2 - x^2)}\dfrac{(1 - x^{2n+2})}{(1 + \lambda x^{2n+2})}\left(\dfrac{r}{a}\right)\cos\theta \\ \qquad + \dfrac{\lambda^2}{(\lambda^2 - x^2)^2}\dfrac{(1 - x^{2n+2})^2}{(1 + \lambda x^{2n+2})^2}\left(\dfrac{r}{a}\right)^2 \end{array}\right\},$$

$$\sigma_{2n}^2 = \left\{\begin{array}{l} \dfrac{1}{(1+\lambda)^2x^2} + 2\dfrac{1}{(1+\lambda)x}\dfrac{1}{(\lambda^2 - x^2)}\dfrac{(1 - \lambda^2x^{2n})}{(1 + \lambda x^{2n})}\left(\dfrac{r}{a}\right)\cos\theta \\ \qquad + \dfrac{1}{(\lambda^2 - x^2)^2}\dfrac{(1 - \lambda^2x^{2n})^2}{(1 + \lambda x^{2n})^2}\left(\dfrac{r}{a}\right)^2 \end{array}\right\},$$

$$\sigma_{2n+1}^2 = \left\{\begin{array}{l} \dfrac{1}{(1+\lambda)^2x^2} + 2\dfrac{1}{(1+\lambda)x}\dfrac{\lambda}{(\lambda^2 - x^2)}\dfrac{(1 - x^{2n+2})}{(\lambda + x^{2n+2})}\left(\dfrac{r}{a}\right)\cos\theta \\ \qquad + \dfrac{\lambda^2}{(\lambda^2 - x^2)^2}\dfrac{(1 - x^{2n+2})^2}{(\lambda + x^{2n+2})^2}\left(\dfrac{r}{a}\right)^2 \end{array}\right\},$$

x and λ being defined by (10) and (12).

3. A much easier problem arises when the two spheres are in contact. Consider the following Fig. 3. We again take the origin at 0, $\overrightarrow{OF'}$ to be $\theta = 0$,

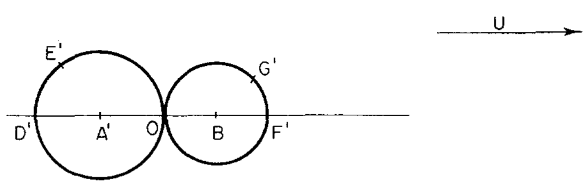

FIG. 3

$\overrightarrow{OD'}$ to be $\theta = \pi$: and the boundary conditions are precisely the same as in § 2. We once more make the transformation (1), this time taking

$$k^2 = ab,$$
$$C = -1,$$

so that the transformation is

$$\begin{aligned} \Theta &= \theta, \qquad R = ab/r, \qquad \Psi = [\sqrt{(ab)}]\, \psi/r, \\ \theta &= \Theta, \qquad r = ab/R, \qquad \psi = [\sqrt{(ab)}]\, \Psi/R. \end{aligned} \tag{20}$$

The transformed figure is Fig. 4 where the sphere $OD'E'$ becomes the plane DE, and the sphere $OF'G'$ the plane FG. We have

$$OD = \tfrac{1}{2}b,$$
$$OF = \tfrac{1}{2}a,$$

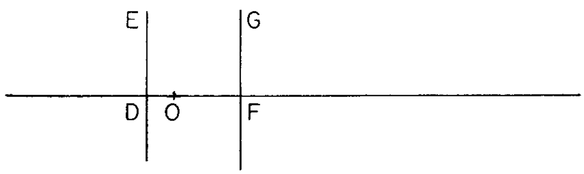

FIG. 4

and the boundary conditions in the transformed problem are (i) $\Psi = 0$ when $\Theta = 0$ and $\Theta = \pi$, (ii) $\Psi = 0$ on the planes DE and FG, (iii)

$$\Psi = -\tfrac{1}{2}(ab)^{3/2}\, U \frac{\sin^2 \Theta}{R} + O\left[R \log\left(\frac{1}{R}\right)\right]$$

when $R \to 0$. Exactly as before, the first term is that due to a doublet of strength $\frac{1}{2}U(ab)^{3/2}$ at the origin: and the problem can again be solved by images.

The images of the doublet at the origin, in the two planes DE and FG, are two equal doublets, of opposite sign, placed respectively at distances b to the left, and a to the right, of O in the figure. These images give rise to two further images, and so on. Let the successive images in the plane DE be at distance $d_0, d_1, d_2, \ldots$ to the left of O, and let the successive images in the plane FG be at distances $e_0, e_1, e_2, \ldots$ to the right of O, each d and e being positive.

Then we have

$$\left.\begin{aligned} d_0 &= b, \\ e_0 &= a, \\ d_{n+1} &= b + e_n\,, \\ e_{n+1} &= a + d_n\,. \end{aligned}\right\} \quad (n \geqslant 0) \tag{21}$$

From (21) we have

$$\left.\begin{aligned} d_{n+2} - d_n &= a + b, \\ e_{n+2} - e_n &= a + b. \end{aligned}\right\} \quad (n \geqslant 0) \tag{22}$$

Once more, it is convenient to solve (22) for the even terms separately, and deduce the odd terms from (21). Thus let

$$\left.\begin{aligned} d_{2n} &= u_n\,, \\ e_{2n} &= v_n\,. \end{aligned}\right\} \quad (n \geqslant 0) \tag{23}$$

Then we have, from (21) and (22), writing $2n$ for n,

$$\begin{aligned} u_{n+1} - u_n &= a + b, \\ v_{n+1} - v_n &= a + b, \\ u_0 &= b, \\ v_0 &= a, \end{aligned}$$

and therefore

$$\left.\begin{aligned} u_n &= na + (n+1)b, \\ v_n &= (n+1)a + nb. \end{aligned}\right\} \quad (n \geqslant 0)$$

From (21) and (23) we therefore have

$$\begin{aligned} d_{2n} &= na + (n+1)b, \\ e_{2n} &= (n+1)a + nb, \\ d_{2n+1} &= (n+1)(a+b), \\ e_{2n+1} &= (n+1)(a+b). \end{aligned} \tag{24}$$

All the image doublets are equal in strength to the one at the origin: those corresponding to even suffices point in the opposite direction to that of the doublet at the origin, and those corresponding to odd suffices, in the same direction. Thus we easily see that the stream function Ψ is

$$\Psi = -\tfrac{1}{2}(ab)^{3/2}\,UR^2\sin^2\Theta$$

$$\left[\frac{1}{R^3} - \sum_{n=0}^{\infty}\frac{1}{\{R^2 + 2[na + (n+1)b]\,R\cos\Theta + [na + (n+1)b]^2\}^{3/2}}\right.$$

$$-\sum_{n=0}^{\infty}\frac{1}{\{R^2 - 2[(n+1)a + nb]\,R\cos\Theta + [(n+1)a + nb]^2\}^{3/2}}$$

$$+\sum_{n=0}^{\infty}\frac{1}{[R^2 + 2(n+1)(a+b)\,R\cos\Theta + (n+1)^2\,(a+b)^2]^{3/2}}$$

$$\left.+\sum_{n=0}^{\infty}\frac{1}{[R^2 - 2(n+1)(a+b)\,R\cos\Theta + (n+1)^2(a+b)^2]^{3/2}}\right]$$

and again it is easy to see that all four series on the right, together with the various series obtained by partial differentiation, are absolutely and uniformly convergent in any bounded domain of R and Θ, corresponding to a region between the planes DE and FG. (This follows by comparison of each series with the series $\sum 1/n^3$.)

Making the transformation (20), back to the original variables, we have the solution to the original problem, namely,

$$\psi = -\tfrac{1}{2}\,Ur^2\sin^2\theta$$

$$\left[1 - \sum_{n=0}^{\infty} 1\Big/\left[1 + 2\left(\frac{n+1}{a} + \frac{n}{b}\right) r\cos\theta + \left(\frac{n+1}{a} + \frac{n}{b}\right)^2 r^2\right]^{3/2}\right.$$

$$-\sum_{n=0}^{\infty} 1\Big/\left[1 - 2\left(\frac{n}{a} + \frac{n+1}{b}\right) r\cos\theta + \left(\frac{n}{a} + \frac{n+1}{b}\right)^2 r^2\right]^{3/2}$$

$$+\sum_{n=0}^{\infty} 1\Big/\left[1 + 2(n+1)\left(\frac{1}{a} + \frac{1}{b}\right) r\cos\theta + (n+1)^2\left(\frac{1}{a} + \frac{1}{b}\right)^2 r^2\right]^{3/2}$$

$$\left.+\sum_{n=0}^{\infty} 1\Big/\left[1 - 2(n+1)\left(\frac{1}{a} + \frac{1}{b}\right) r\cos\theta + (n+1)^2\left(\frac{1}{a} + \frac{1}{b}\right)^2 r^2\right]^{3/2}\right]$$

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